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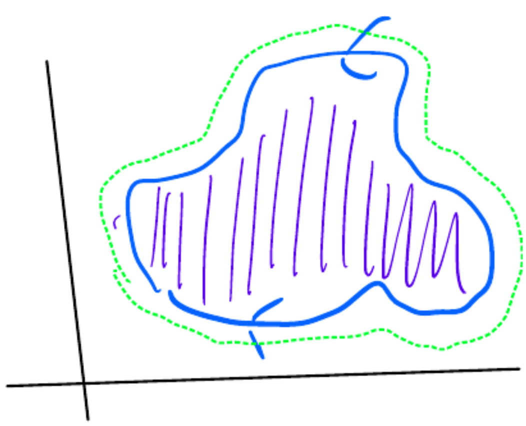
Last Time: Green's Theorem.

Prop (Green's Theorem): Suppose D is a region in the plane with it's boundary a piece wise smooth simple-closed curve. If $P(x,y)$ and $Q(x,y)$ have cts partial derivatives on some open region "R", containing D, then

$$\boxed{\int_{\partial D} P\,dx + Q\,dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$

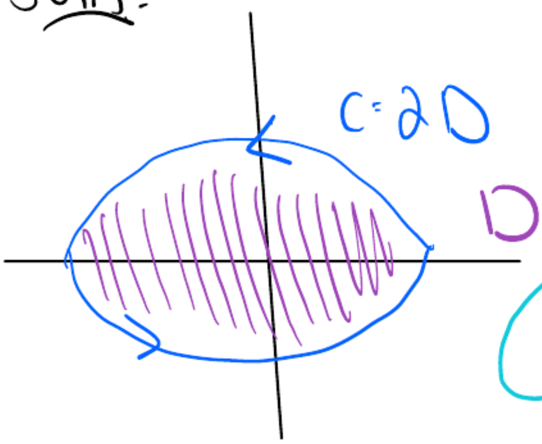


Ex: Compute $\int_C y^4\,dx + 2xy^2\,dy$ for C the positively oriented ellipse $x^2+2y^2=2$.

notice: like a circle. so make a substitution to make it circle like then use polar coords. (Transformation)

$$\begin{cases} x = \sqrt{2}\, r\cos\theta \\ y = r\sin\theta \end{cases}$$

Soln:



$*$ $0 \le r \le 1$ $*$

$0 \le \theta \le 2\pi$

By Green's Theorem:

$$\int_C y^4\,dx + 2xy^2\,dy$$

$$= \iint_D \left(\frac{\partial}{\partial x}\left[2xy^2\right] - \frac{\partial}{\partial y}\left[y^4\right] \right) dA$$

$$= \iint_D \left(2y^2 - 4y^3\right) dA$$

Notice cont:

unit disk

$$x^2+2y^2=2$$

iff $(\sqrt{2}\, r\cos(\theta))^2+2(r\sin(\theta))^2=2$

iff $2r^2\cos^2(\theta)+2r^2\sin^2(\theta)=2$

iff $2r^2((\cos^2\theta)+(\sin^2\theta))=2$

iff $2r^2(\cos^2(\theta)+\sin^2(\theta))=2$

iff $r^2=1$ iff $\underline{r=1 \text{ and } r\geq 0}$

Now compute Jacobian:

$$\frac{\partial(x,y)}{\partial(r,\theta)}=\det\begin{bmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{bmatrix}$$

$$= \det \begin{bmatrix} \sqrt{2}\cos\theta & -\sqrt{2}\, r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$= \sqrt{2}\, r \overbrace{\cos^2\theta \underset{(- \quad -)}{+} \sqrt{2}\, r \sin^2}^{1}\theta$$

$$= \sqrt{2}\, r$$

$$\therefore \int_C y^4\,dx + 2xy^2\,dy = \iint_D (2y^2 - 4y^3)\,dA$$

$$= \int_{r=0}^{1} \int_{\theta=0}^{2\pi} \left(2(r\sin\theta)^2 - 4(r\sin\theta)^3\right)\sqrt{2}\, r \; d\theta\, dr$$

$$= \int_{r=0}^{1} \int_{\theta=0}^{2\pi} 2\sqrt{2}\, r^3 \left(\sin^2\theta - 2r\sin^3\theta\right) d\theta\, dr$$

$$= \int_{r=0}^{1} 2\sqrt{2}\, r^3 \int_{\theta=0}^{2\pi} \sin^2\theta\,(1 - 2r\sin(\theta))\, d\theta\, dr$$

$$= \int_{r=0}^{1} 2\sqrt{2}\, r^3 \int_{\theta=0}^{2\pi} (1-\cos^2\theta)(1-2r\sin\theta\, d\theta\, dr$$

<u>eval. inner integral:</u>

$$\int_{\theta=0}^{2\pi} (1-\cos^2\theta)(1-2r\sin(\theta)\, d\theta$$

$$= \int_{\theta=0}^{2\pi} (1-\cos^2\theta)\, d\theta - 2r \int_{\theta=0}^{2\pi} (1-\cos^2\theta)\sin(\theta) d\theta$$

$u = \cos\theta$
$du = \sin\theta\, d\theta$

$$= \int_{\theta=0}^{2\pi} (1-\tfrac{1}{2}(1+\cos 2\theta)\, d\theta - 2r \int -(1-u^2)\, du$$

$$= \int_{\theta=0}^{2\pi} \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\right) d\theta + 2r\left[u - \tfrac{1}{3}u^3\right]_{\theta=0}^{2\pi}$$

$$= \left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_{\theta=0}^{2\pi} + 2r\left[\cos\theta - \tfrac{1}{3}(\cos^3\theta)\right]_{\theta=0}^{2\pi}$$

$$= \tfrac{1}{2}(2\pi - 0) - \tfrac{1}{4}(\sin(4\pi) - \sin(0) + 2r((\cos(2\pi) - \cos(0)) - \tfrac{1}{3}(\cos^3 2\pi - \cos^3(0))))$$

$$= \pi - \tfrac{1}{4}\cdot 0 + 2r(0 - \tfrac{1}{3}\cdot 0) = \pi$$

eval outer integral:

$$\int_{r=0}^{1} 2\sqrt{2}\, r^3 \pi \, dr = \frac{2\sqrt{2}\pi}{4}\left[r^4\right]_{r=0}^{1}$$

$$= \frac{\pi}{\sqrt{2}}\left(1^4 - 0^4\right) = \frac{\pi}{\sqrt{2}}$$

$$\therefore \int_C y^4\,dx + 2xy^2\,dy = \boxed{\frac{\pi}{\sqrt{2}}}$$

NB: So far, all examples so far have turned line integrals into double integrals via Greene's Theorem. BUT we can go in other way as well.

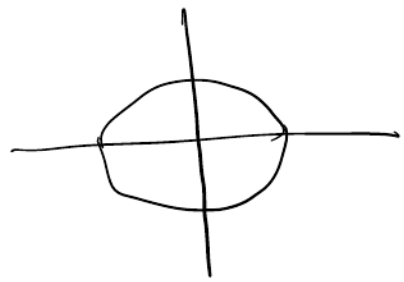
NB: If P Q satisfy $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$,

then via Green's theorem

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D 1\,dA = \text{Area}(D)$$

Ex: Compute the area of the general ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$



Soln: $\text{Area}(D) = \int_{\partial D} P\,dx + Q\,dy$

if we choose

P, Q w/ $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$

e.g. $Q = 0$, $P = -y$: $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}[0] - \frac{\partial}{\partial y}[-y] = 1$

$\therefore$ area $(D) = \int_D -y\,dx + 0\,dy = -\int_{\partial D} y\,dx$

find ∂D:

the ellipse ∂D is parametrized by

$\vec{r}(t) = \langle a\cos(t), b\sin(t) \rangle$ on

$0 \leq t \leq 2\pi$ $\quad \therefore \; dx = x'(t)\,dt$

$$= -\int_{t=0}^{2\pi} b\sin(t) \cdot -a\sin(t)\,dt$$

$$= ab\int_{t=0}^{2\pi} \sin^2(t)\,dt$$

$$= ab\int_{t=0}^{2\pi} \frac{1}{2}(1-\cos(2t))\,dt$$

$$= \frac{1}{2}ab\left[t - \frac{1}{2}\sin(2t)\right]_{t=0}^{2\pi}$$

$$= \frac{1}{2}ab\left((2\pi-0) - \frac{1}{2}(0-0)\right) = \boxed{ab\pi}$$

§16.5: Curl and Divergence

Goal: Define and study two new operations on vector fields.

Curl: The curl of vector field $\vec{v}$ on $\mathbb{R}^3$ is

$\vec{v} = \langle P, Q, R \rangle$

$$\text{curl}(\vec{v}) = \text{"} \nabla \times \vec{v} \text{"} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle P, Q, R \rangle$$

$$= \det \begin{bmatrix} \vec{i} & \vec{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix}$$

$$= \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, -\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right), \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

Ex: Compute curl$(\vec{v})$ for $\vec{v} = \langle xy, xyz, -y^2 \rangle$

Soln:

$$\text{curl}(\vec{v}) = \nabla \times \vec{v} = \det \begin{bmatrix} \vec{i} & \vec{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xyz & -y^2 \end{bmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}\left[-y^2\right] - \frac{\partial}{\partial z}\left[xyz\right], -\left[\frac{\partial}{\partial x}\left[-y^2\right] - \frac{\partial}{\partial z}\left[xy\right]\right], -\frac{\partial}{\partial y}\left[xy\right] \right\rangle$$

$$= \langle -2y - xy, -(0-0), yz - x \rangle$$

$$= \langle -xy - 2y, 0, yz - x \rangle$$

Observation: Suppose $\vec{v} = \nabla f$ is conservative.

i.e. $\vec{v} = \langle f_x, f_y, f_z \rangle$

So, $\text{curl}(\vec{v}) = \nabla \times \vec{v} = \det \begin{bmatrix} \vec{i} & \vec{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{bmatrix}$

$$= \left\langle \frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial f_z}, -\left(\frac{\partial f_z}{\partial x} - \frac{\partial f_x}{\partial f_z}\right), \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right\rangle$$

$= \langle f_{zy} - f_{yz}, -(f_{zx} - f_{xz}), f_{yx} - f_{xy} \rangle$

$= \langle 0, 0, 0 \rangle = \vec{0}$ by Clairaut's theorem.

Point: $\text{Curl}(\nabla f) = \vec{0}$, i.e. Curl of a conservative ∇f is $\vec{0}$.

Prop: A vector field w/ components having cts. partial derivatives is conservative if and only if $\text{Curl}(\vec{v}) = \vec{0}$.

<u>Divergence</u>: The divergence of vector field $\vec{v}$, $\vec{v} = \langle v_1, v_2, \dots, v_n \rangle$ is

$$\text{div}(\vec{v}) = \text{"}\nabla \cdot \vec{v}\text{"} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x_2}, \dots \frac{\partial}{\partial x_n} \right\rangle \cdot \langle v_1, v_2, \dots, v_n \rangle$$

$$= \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \dots + \frac{\partial v_n}{\partial x_n}$$

Ex: For $\vec{v} = \langle xy, xyz, -y^2 \rangle$ compute
div $(\vec{v})$.

Sol. div $(\vec{v}) = \frac{\partial}{\partial x}[xy] + \frac{\partial}{\partial y}[xyz] + \frac{\partial}{\partial z}[-y^2]$

$= y + xz + 0 = y + xz$

Suppose: $\vec{v} = \text{curl}(\vec{w})$ for $w = \langle P, Q, R \rangle$

$\vec{v} = \langle R_y - Q_z, -(R_x - P_z), Q_x - P_y \rangle$

now, div $(\vec{v}) = \frac{\partial}{\partial x}[R_y - Q_z] + \frac{\partial}{\partial y}[-(R_x - P_z)] + \frac{\partial}{\partial z}[Q_x - P_y]$

$= (P_{zy} - P_{yz}) + (Q_{xz} - Q_{zx}) + (R_{yx} - R_{xy})$

$= 0 + 0 + 0 = 0$ by clairaut's theorm.

Prop: A vector field is the curl of another vector field if and only if its divergence is zero.

NB: above we showed $\mathrm{div}(\mathrm{curl}(\vec{\omega})) = 0$.